Casa abierta al tiempo
Universidad Autónoma Metropolitana

<u>Dr. José Raúl Montes de Oca Machorro</u>
Jefe del Departamento
División de Ciencias Básicas e Ingeniería

C.B.I.MAT.071.2025
31 de julio 2025

Dr. Román Linares Romero
Presidente del Consejo Divisional
División de Ciencias Básicas e Ingeniería
P r e s e n t e

Por medio del presente me permito solicitar, se incluya en el Orden del Día de la próxima Sesión del Consejo Divisional, el informe del periodo sabático que presenta el **Dr. Josue Meléndez Sánchez (29367)**.

Agradeciendo la atención a la presente, quedo a sus órdenes para cualquier aclaración que requiera al respecto.

A t e n t a m e n t e
"Casa Abierta al Tiempo"
[redacted]

Anexo: Informe.
Probatorios

DEPARTAMENTO DE MATEMÁTICAS
Av. Ferrocarril San Rafael Atlixco, Núm. 186, Col. Leyes de Reforma 1 A Sección, Alcaldía Iztapalapa, C.P. 09310, Ciudad de México.
Tels. 55-5804-4805,06 y 07
dmat@xanum.uam.mx, www.izt.uam.mx

UNIVERSIDAD AUTÓNOMA METROPOLITANA

Casa abierta al tiempo

CONSEJO DIVISIONAL DE CIENCIAS BÁSICAS E INGENIERIA

INFORME DE PERÍODO SABÁTICO

DATOS GENERALES

Nombre del profesor: Josué Meléndez Sánchez   Nº empleado: 29367

Departamento: Matemáticas   Área: Ecuaciones Dif. y Geometría

Teléfono particular: [redacted] Extensión UAM-I: ______ E-mail [redacted] @xanum.uam.mx

DATOS DEL PERÍODO SABÁTICO SOLICITADO

Nº meses solicitados: 12   Fecha de inicio: 8 Julio 2024   Fecha de terminación: 7 Julio 2025

Institución donde se realizará: ______

Depto., Laboratorio, etc.: Departamento de Matemáticas, UAM-Iztapalapa

Domicilio de la institución: Av. San Rafael Atlixco 186, Col. Vicentina, 09349, CDMX

Teléfono [redacted] Fax: ______ E-ma [redacted] @xanum.uam.mx

OBJETIVOS DEL PERÍODO SABÁTICO

- Realizar investigación en la teoría de subvariedades.
- Difundir la investigación en eventos nacionales o internacionales.
- Continuar con la dirección de un estudiante de posgrado.

METAS ALCANZADAS EN EL PERÍODO SABÁTICO

- [ ] Memorias in extenso en libro de resúmenes*
- [ ] Artículos de investigación en revista indexada*
- [x] Presentaciones en congresos
- [ ] Libros o capítulos de libros
- [ ] Grado
- [ ] % Avance de estudios de posgrado
- [x] Otros (especifique): Se sometieron 2 artículos de investigación.

* Indicar en anexo si se trata de trabajo publicado, aceptado o sometido

TIPO DE ACTIVIDADES ACADÉMICAS DESARROLLADAS

(Indique aquellas relacionadas con las actividades desarrolladas)

- [x] Investigación
- [ ] Docencia
- [x] Difusión
- [x] Formación académica
- [ ] Formación profesional
- [ ] Entrenamiento técnico
- [x] Otros (especifique): Participación en el Instituto Carlos Graef, Conferencias de divulgación e investigación.

RESUMEN DEL PLAN DE ACTIVIDADES ACADÉMICAS DESARROLLADAS

(El llenado de esta sección no sustituye el informe detallado de actividades)

1. Dirección de una tesis de un estudiante de doctorado.
2. Elaboración de dos artículos de investigación y envió a revistas con arbitraje.
3. Participación y difusión en eventos académicos internos y externos a la UAM.
4. Dirección de un proyecto de un alumno de licenciatura en matemáticas.
5. Arbitraje de un artículo de investigación. Revisor de un libro de matemáticas de nivel de licenciatura.
6. Revisión de una tesis de doctorado y sinodal del examen de grado del mismo.

PARA USO DEL JEFE DE DEPARTAMENTO

Después de haber evaluado el informe detallado de actividades del período sabático del interesado según los lineamientos establecidos para tal efecto; informo al Consejo Divisional que:

- (X) Los objetivos SE cumplieron satisfactoriamente
- ( ) Los objetivos SE cumplieron parcialmente
- ( ) Los objetivos NO se cumplieron
- ( ) NO se cumplió el propósito del sabático

[redacted]

Firma del Jefe de Departamento

31/Julio/2025

Fecha

PARA USO DEL CONSEJO DIVISIONAL

El Consejo Divisional, en su Sesión No. ______________ del ________________________________ sobre el Período sabático del interesado acordó que:

- ( ) Los objetivos SE cumplieron satisfactoriamente
- ( ) Los objetivos SE cumplieron parcialmente
- ( ) Los objetivos NO se cumplieron
- ( ) NO se cumplió el propósito del sabático

____

Secretario del Consejo Divisional

*Además de este formato-resumen, el interesado deberá entregar su Informe detallado de actividades junto con la documentación probatoria correspondiente.

Informe de actividades del período sabático

8 de julio de 2024 a 7 de julio de 2025

Dr. Josué Meléndez Sánchez

Las actividades que se realizaron en el período sabático son las siguientes:

1. Dirección del Proyecto de Investigación I del alumno Benjamín Ozmar Hernández Alvarado de la Licenciatura en Matemáticas durante el trimestre 24-P. Se envió una propuesta del proyecto al Comité de la Licenciatura sobre los temas a trabajar. Se dirigió el trabajo de Benjamín personalmente con asesorías semanales. Se obtuvo un primera versión sobre los temas de variedades en el espacio euclidiano.

2. Dirección del Trabajo de Investigación IV del alumno de doctorado Eduardo Rodríguez Romero durante el trimestre 24-P. En este trimestre, en forma conjunta, se escribió el articulo de investigación "*O(n)*x*O(m)-invariant hypersurfaces with constant mean curvature in Euclidean space*" sobre el comportamiento del cuadrado de la norma de la segunda forma fundamental asociado a una familia de hipersuperficies invariantes bajo la acción de un producto de grupos ortogonales inmersas en un espacio euclidiano.

3. Arbitraje del artículo de investigación con número JMAA-24-1668 "*Local rigidity of constant mean curvature hypersurfaces in space forms*" para la revista Journal of Mathematical Analysis and Applications.

4. Revisión del articulo de investigación *Gromov's tori are optimal*, Anton Petrunin, Geometric and Functional Analysis, vol. 34 (2024) No. 1, págs. 202–208, para la base de datos Mathscinet.

5. Impartición de la plática *Sobre la norma de la segunda forma fundamental de hipersuperficies*. Impartido el 20 de septiembre de 2024 en Diálogos Virtuales de Análisis y Geometría (DIVAGEO) del Departamento de Matemáticas de la Facultad de Ciencias, UNAM.

6. Sinodal: Conclusión de la revisión de una tesis de doctorado de la UNAM del alumno José Eduardo Nuñez Ortiz, titulado "*Caracterización de superficies paralelas en el espacio de Minkowski*".

7. Proyecto de Investigación II de licenciatura del alumno Benjamín Ozmar Hernández Alvarado. El alumno ha finalizando su trabajo "*Una introducción a la transversalidad*" y enviado a la Comisión de la Licenciatura de Matemáticas con un informe final. Benjamín también presentó su proyecto en el Departamento de Matemáticas, dentro de las exposiciones de los proyectos de Licenciatura. Esto se realizó durante el trimestre 24-O. Por ser un trabajo extenso, se adjunta únicamente la carátula, el índice y la introducción del trabajo.

8. Trabajo de Investigación V del alumno de doctorado Eduardo Rodríguez Romero en el trimestre 24-O: En este trimestre fue aceptado el artículo de investigación: "*A note on surfaces with constant angle intersection in Riemannian manifolds*" en el Boletín de la Sociedad Matemática Mexicana, se adjunta carátula del artículo, aún cuando la fecha de envío fue antes del periodo sabático. También se sometió el artículo de investigación "*O(n)*x*O(m)-invariant hypersurfaces with constant mean curvature in Euclidean space*".

9. Impartición de la plática presencial sobre: *Superficies con intersección de ángulo constante en variedades riemannianas*. XIX Coloquio de geometría en la Facultad de Matemáticas de la Universidad Autónoma de Yucatán, UADY. Celebrado el 11 de diciembre de 2024.

10. Impartición de la plática presencial: *Estimación de la norma de la segunda forma fundamental de una hipersuperficie con CMC*. 57º Congreso Nacional de la Sociedad Matemática Mexicana en la Facultad de Ciencias Exactas de la Universidad Juárez del Estado de Durango, llevado a cabo del 21 al 25 de octubre del 2024

11. Impartición del taller: *Análisis geométrico de superficies: una introducción elemental,* en el 7º Coloquio del Departamento de Matemáticas UAM del 27 al 31 de enero de 2025.

12. Conferencia: *Explorando el teorema de Gauss-Bonnet,* dentro de las actividades del Instituto Carlos Graef, Jóvenes hacia la Ciencia y la Ingeniería. Impartida el 8 de marzo de 2025, UAM-I.

13. Conferencia: *Subvariedades en variedades riemannianas,* en el Seminario del área de Ecuaciones Diferenciales y Geometría del departamento de matemáticas de la UAM-I. Impartida el 19 de marzo 2025.

14. Trabajo de Investigación VI (doctorado) del alumno Eduardo Rodríguez Romero durante el trimestre 25-I: Los árbitros respondieron al artículo de investigación enviado. Se hicieron las correcciones y además se propuso una extensión del teorema principal. Se sometió a una segunda revisión a la misma revista. Aún se espera la tercera revisión de los árbitros. Se adjunta primera hoja del envío del manuscrito como comprobante.

15. Participación como miembro del Comité de la Licenciatura en Matemáticas.

16. Sinodal de examen doctoral de alumno externo: José Eduardo Núñez Ortiz (UNAM). Tesis: *Caracterización de superficies paralelas en el espacio de Minkowski*. Fecha de examen: viernes 14 de marzo de 2025.

17. Revisión del libro de la UNAM: *Una introducción a la geometría lorentziana,* en el período de febrero a junio de 2025.

18. Revisión del articulo de investigación: *A Simons-type integral inequality for minimal surfaces with constant Kähler angle in complex projective spaces* de los autores Fei, Jie and Jiao, Xiaoxiang and Wang, Jun, Frontiers of Mathematics, vol. 19, (2024), no. 6, 1007–1024, para la base de datos Mathscinet.

19. Conferencia: *¿Qué es la geometría diferencial?* Impartida en el Colegio de Bachilleres el 11 de junio de 2025. Dicha conferencia se realizó en forma presencial en el plantel número 7.

20. Evaluador de una solicitud de la Convocatoria 2025 *Estancias Sabáticas Vinculadas a la Consolidación de Grupos de Investigación* por parte del SECIHTI.

21. Comencé a impartir el curso de maestría *Geometría Diferencial y Riemanniana* en el trimestre 25-P.

22. Envío del articulo de investigación titulado *Some remarks on warped products*. Trabajo en conjunto con mi alumno de doctorado Eduardo Rodríguez Romero. Se adjunta el manuscrito como comprobante.

23. Se impartió en forma presencial dos sesiones de un taller de geometría diferencial, llamado *Explorando el Tensor Métrico,* dirigido a estudiantes de la Licenciatura de Matemáticas y Física de la UAM-I. Fruto de este trabajo fueron publicados 2 videos en mi canal de *YouTube* para un alcance aún mayor a otros estudiantes interesados en los temas.

    - Sesión 1. Publicado el 9 septiembre 2024 y cuenta con más de 1,900 vistas a la fecha de conclusión del sabático.
    - Sesión 2. Publicado el 18 marzo 2025 y cuenta con más de 2,100 vistas a la fecha de conclusión del sabático.
    - Los enlaces son:
      Sesión 1: https://www.youtube.com/watch?v=Vto9woDLOAM
      Sesión 2: https://www.youtube.com/watch?v=KPg4q-3kH9A&t=288s

[redacted]

Dr. Josué Meléndez Sánchez

JMAA-24-1668 Review Completed

[redacted]

Fecha 2024-08-08 13:07

Ms. No.: **JMAA-24-1668**
Title: Local rigidity of constant mean curvature hypersurfaces in space forms
Corresponding Author: Professor Tongzhu Li
Authors: Yayun Chen; Tongzhu Li

Dear Dr. Meléndez Sánchez,

We would like to thank you for your participation in the evaluation of the above manuscript.

We rely on reviewers like you, unselfishly giving up time to help us determine the suitability of a manuscript. We appreciate the time that you have contributed to this important component of the peer review process.

Should you need to access your review comments, please log onto the Editorial Manager at:
[redacted]

If you do not know your confidential password, you may reset it by clicking this [redacted]

As a token of appreciation, we would like to provide you with a review recognition certificate on Elsevier Reviewer Hub (reviewerhub.elsevier.com). Through the Elsevier Reviewer Hub, you can also keep track of all your reviewing activities for this and other Elsevier journals on Editorial Manager.

If you have not yet activated your 30 day complimentary access to ScienceDirect and Scopus, you can still do so via the [Rewards] section of your profile in Reviewer Hub (reviewerhub.elsevier.com).
You can always claim your 30-day access period later, however, please be aware that the access link will expire six months after you have accepted to review.

Kind regards,

Oscar Palmas, Dr.
Associate Editor
Journal of Mathematical Analysis and Applications

#REV_JMAA#

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JMAA-24-1668R2 Reviewer Notification of Editor Decision

[redacted] [redacted]

Fecha 2024-10-14 15:07

Ref: **JMAA-24-1668R2**
Title: Local rigidity of constant mean curvature hypersurfaces in space forms
Article Type: Regular Article

Dear Dr. Meléndez Sánchez,

Thank you once again for reviewing the above-referenced paper. With your help the following final decision has now been reached:

Accept

We appreciate your time and effort in reviewing this paper and greatly value your assistance as a reviewer for Journal of Mathematical Analysis and Applications.

Yours sincerely,

Steven Krantz, PhD
Editor-in-Chief
Journal of Mathematical Analysis and Applications

****
For further assistance, please visit our customer support site at [redacted]
Here you can search for solutions on a range of topics, find answers to frequently asked questions and learn more about EM via interactive tutorials. You will also find our 24/7 support contact details should you need any further assistance from one of our customer support representatives.

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[redacted]

Publications results for "Items reviewed by Meléndez, Josué"

MR4706446 Reviewed

Petrunin, Anton (1-PAS)
Department of Mathematics, Pennsylvania State University
University Park (State College), Pennsylvania, 16802

Gromov's tori are optimal. (English summary)

Geom. Funct. Anal. 34 (2024), no. 1, 202–208.
53C42

Let $\mathbb{B}^q$ be a closed unit ball in $\mathbb{R}^q$ and $\mathbb{T}^n$ the n-dimensional torus. Motivated by examples of embeddings $\mathbb{T}^n \hookrightarrow \mathbb{R}^q$ for large q, the author proves that if $\mathbb{T}^n$ is smoothly immersed in $\mathbb{B}^q$ then its maximal normal curvature is at least

$$\sqrt{3 \cdot \frac{n}{n+2}}.$$

Reviewed by Josué Meléndez

References

1. Aubin, T.: Nonlinear Analysis on Manifolds. Monge–Ampère Equations. Grundlehren der Mathematischen Wissenschaften., vol. 252 (1982). MR0681859
2. Burago, Y.D., Zalgaller, V.A.: Geometric Inequalities. Grundlehren der Mathematischen Wissenschaften., vol. 285 (1988). MR0936419
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4. Gromov, M.: Curvature, Kolmogorov diameter, Hilbert rational designs and overtwisted immersions (2022a). arXiv:2210.13256 [math.DG].
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10. Petrunin, A., Zamora Barrera, S.: What Is Differential Geometry: Curves and Surfaces (2022). arXiv:2012.11814 [math.HO].
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12. Tabachnikov, S.: The tale of a geometric inequality. In: MASS selecta: teaching and learning advanced undergraduate mathematics pp. 257–262 (2003). MR2027183

This list reflects references listed in the original paper as accurately as possible with no attempt to correct error.

DIVAGEO

Diálogos Virtuales de Análisis y Geometría

El Departamento de Matemáticas de la Facultad de Ciencias
otorga la presente

CONSTANCIA

A: **Josué Meléndez Sánchez**

por impartir la conferencia

“Sobre la norma de la segunda forma fundamental de hipersuperficies”

el día 20 de septiembre del 2024

[redacted]

Dr. Juan Carlos Fernández *Dr. Jesús Núñez Zimbrón* *Dr. Oscar Palmas*

Casa abierta al tiempo
UNIVERSIDAD AUTÓNOMA METROPOLITANA
Unidad Iztapalapa

Departamento de Matemáticas

Introducción a la transversalidad de subvariedades

Proyectos de Investigación I y II

Ozmar Benjamín Hernández Alvarado

Matrícula: [redacted]

Asesor: Dr. Josué Meléndez Sánchez

24 de enero de 2025

Índice general

Introducción

En este trabajo se mostrarán varias definiciones y conceptos necesarios para que el lector pueda comprender los conceptos básicos de la topología diferencial en el espacio euclidiano. En particular, se introducirán algunas propiedades de suavidad de funciones, las cuales resultan de gran relevancia para el desarrollo posterior de la teoría.

Se incluirán también ejemplos concretos que ayudarán a ilustrar la aplicación de los conceptos y técnicas desarrollados, proporcionando una visión más clara y accesible de la teoría. Algunos de ellos fueron tomados y resueltos de los ejercicios del libro de Guillemin, V. y Pollack.

Bol. Soc. Mat. Mex. (2025) 31:24
https://doi.org/10.1007/s40590-024-00704-6

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

A note on surfaces with constant angle intersection in Riemannian manifolds

Josué Meléndez[1] · **Eduardo Rodríguez-Romero**[1]

Received: 18 April 2024 / Accepted: 5 December 2024

Abstract
We study ruled surfaces that intersect with constant angle a surface M in a Riemannian manifold $\overline{M}^3$, thus extending some results recently given in (Meléndez and Rodríguez-Romero in Differ Geom Appl 91:102063, 2023). We also deal with normal and tangent submanifolds of arbitrary dimension, giving some properties about them.

Keywords Ruled surface · Constant angle intersection · Mean curvature · Gauss curvature · Darboux frame

Mathematics Subject Classification 53C40 · 53C42

1 Introduction

Research on constant angle hypersurfaces is wide and of current interest both in the Riemannian and semi-Riemannian contexts, see for instance [1, 3, 6, 7] and references therein. Recall that a constant angle hypersurface M satisfies that its normal vector η makes a constant angle with some fixed vector field X of the ambient space $\overline{M}$.

This notion can be relaxed by asking that the constant angle property is satisfied only along a submanifold of M. Examples of this can be found in [4, 5], where the authors studied normal and tangent hypersurfaces M and N in a Riemannian manifold $\overline{M}^n$. If we denote by η and ξ the normal vectors of M and N, respectively, then normality (tangency) of M and N are equivalent to the condition $\angle(\eta, \xi) = \phi_0$ along $M \cap N$, with $\phi_0 = \frac{\pi}{2}$ ($\phi_0 \in \{0, \pi\}$). Therefore, the ruled surfaces studied in [5] are particular cases of the following notion of constant angle intersection.

Josué Meléndez was partially supported by programa especial de apoyo a proyectos de docencia e investigación de la UAM: *"Ecuaciones diferenciales y geometría diferencial con aplicaciones"*.

✉ Josué Meléndez
jms@xanum.uam.mx

Eduardo Rodríguez-Romero
err29072019@gmail.com

1 Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, CP 09340 México City, México

Published online: 21 December 2024

Birkhäuser

XIX Coloquio de Geometría

Se otorga la presente

CONSTANCIA

a

Josué Meléndez Sánchez

por haber impartido la conferencia "*Superficies con intersección de ángulo constante en variedades riemannianas*" en el marco de las actividades del XIX Coloquio de Geometría, celebrado en las instalaciones de esta Facultad.

Mérida, Yucatán a 11 de diciembre de 2024.

[redacted]

Dr. Carlos Francisco [redacted]o Loeza
Jefe de la Unidad de Posgrado e Investigación

9 - 11 diciembre, 2024

Facultad de Matemáticas

Universidad Autónoma de Yucatán

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La Sociedad Matemática Mexicana

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RECONOCIMIENTO

a: **Josué Meléndez Sánchez**

por la presentación de la conferencia presencial:

Estimación de la norma de la segunda forma fundamental de una hipersuperficie con CMC

en el área de Geometría Diferencial, realizada dentro de las actividades del 57 Congreso Nacional de la Sociedad Matemática Mexicana, llevado a cabo del 21 al 25 de octubre del 2024, en la Facultad de Ciencias Exactas de la Universidad Juárez del Estado de Durango

Octubre 2024

[redacted]

Dra. Gabriela Araujo Pardo
Presidenta de la Junta Directiva

LVII - P - 5973 - 1990

El Departamento de Matemáticas de la División de Ciencias Básicas e Ingeniería de la Universidad Autónoma Metropolitana, Unidad Iztapalapa, otorga el presente

Reconocimiento

a

Josué Melendez Sánchez

por la impartición del **Taller:**

Análisis geométrico de superficies: una introducción elemental

en el marco del 7º Coloquio del Departamento de Matemáticas, realizado del 27 al 31 de enero de 2025.

[redacted]

Dr. José Raúl Montes de Oca Machorro
Jefe del Departamento de Matemáticas

Dr. Mario Pineda Ruelas
Representante del Comité Organizador del Coloquio

7º COLOQUIO DEL DEPARTAMENTO DE MATEMÁTICAS

del 27 al 31 de enero del 2025, Unidad Iztapalapa de la UAM, Ciudad de México

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Departamento de Matemáticas UAM IZTAPALAPA

Posgrado de Matemáticas UAM IZTAPALAPA

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2025 Instituto Carlos Graef 14 años guiando jóvenes hacia la ciencia y la ingeniería

Los Departamentos de Física, Matemáticas, Ingeniería Eléctrica, Ingeniería de Procesos e Hidráulica y Química de la División de Ciencias Básicas e Ingeniería

Otorgan la presente

CONSTANCIA

a

Josué Meléndez Sánchez

Por su participación en las actividades del Instituto Carlos Graef-Colegios de Bachilleres, Jóvenes hacia la Ciencia y la Ingeniería, con la conferencia magistral: Explorando el teorema de Gauss-Bonnet impartida el 8 de marzo de 2025, Ciudad de México.

[redacted]

Dr. Román Linares Romero
Director de División CBI

Casa abierta al tiempo

UNIVERSIDAD AUTÓNOMA METROPOLITANA
Unidad Iztapalapa

EL DEPARTAMENTO DE MATEMÁTICAS OTORGA EL PRESENTE RECONOCIMIENTO AL

DR. JOSUÉ MELÉNDEZ SÁNCHEZ

POR SU CONFERENCIA

SUBVARIEDADES EN VARIEDADES RIEMANNIANAS

EL 19 DE MARZO DEL 2025 EN EL SEMINARIO DE ECUACIONES DIFERENCIALES Y GEOMETRÍA EN ESTE DEPARTAMENTO

ATENTAMENTE

[redacted]

UAM CBI
División de Ciencias Basicas e Ingeniería

DR. JOSÉ RAÚL MONTES DE OCA MACHORRO

JEFE DEL DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD AUTÓNOMA METROPOLITANA - IZTAPALAPA

Rendiconti del Circolo Matematico di Palermo Series 2

$O(n)\times O(m)$-invariant hypersurfaces with constant mean curvature in Euclidean space

--Manuscript Draft--

Manuscript Number:	RCMP-D-24-00688R1
Full Title:	$O(n)\times O(m)$-invariant hypersurfaces with constant mean curvature in Euclidean space
Article Type:	Original Research
Corresponding Author:	Eduardo Rodriguez-Romero, M.Sc. Metropolitan Autonomous University: Universidad Autonoma Metropolitana Mexico City, Mexico City MEXICO
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Metropolitan Autonomous University: Universidad Autonoma Metropolitana
Corresponding Author's Secondary Institution:	
First Author:	Eduardo Rodriguez-Romero, M.Sc.
First Author Secondary Information:	
Order of Authors:	Eduardo Rodriguez-Romero, M.Sc.
	Josué Meléndez, PhD
Order of Authors Secondary Information:	
Funding Information:	
Abstract:	We prove some estimates for the infimum and supremum of the squared norm of the second fundamental form of $O(n)\times O(m)$-invariant hypersurfaces in $\mathbb{R}^{n+m}$ with constant mean curvature $H>0$.
Response to Reviewers:	We are sending you a revised version of our article. We have taken into account all the suggestions of the reviewer. We would greatly appreciate it if our article was considered.

Cd. Universitaria a 3 de julio de 2025

Dr. Josué Meléndez Sánchez

Presente.

A nombre de la Comisión Editorial del Departamento de Matemáticas le agradecemos su valiosa colaboración en la revisión y dictamen de un libro sometido para su arbitraje a las Prensas de Ciencias.

Sabemos el tiempo y esfuerzo que requiere hacer una revisión íntegra de un libro de texto, por lo que agradecemos la responsabilidad demostrada por usted en esta tarea. Sus aportaciones permitirán a esta Comisión Editorial determinar el fallo definitivo y, en caso de ser publicada la obra, su dictamen garantizará el nivel académico del texto.

Le reiteramos nuestro agradecimiento por su esfuerzo, el cual permite que la tarea editorial de nuestra Facultad sea de calidad y pertinencia.

Reciba un cordial saludo.

A T E N T A M E N T E

[redacted]

Por la Comisión Editorial del Departamento de Matemáticas
Dr. Vinicio Antonio Gómez Gutiérrez

[redacted]

Publications results for "Items authored by Fei, Jie"

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A Simons-type integral inequality for minimal surfaces with constant Kähler angle in complex projective spaces. (English summary)

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53C42 (53C55)

Let M be a compact Riemann surface without boundary and $f\colon M \to \mathbb{C}\mathrm{P}^n$ a conformal minimal immersion neither holomorphic nor antiholomorphic with constant Kähler angle. The authors establish a Simons-type integral inequality for M (Theorem 3.4). In this paper, the main result is to determine all the closed minimal surfaces with the square norm of the second fundamental form satisfying a pinching condition. Precisely, they prove the following theorem.

Main Theorem. Let M be a compact Riemann surface without boundary and $f\colon M \to \mathbb{C}\mathrm{P}^n$ be a conformal minimal immersion neither holomorphic nor antiholomorphic. If its Kähler angle θ is constant and the square norm S of the second fundamental form satisfies the pinching condition

$$\frac{3}{4}S^2 - (1 + 2\cos^2\theta)S + 15\cos^2\theta\sin^2\theta - 8\kappa \le 0$$

on M, where κ is a globally defined invariant relative to the first and second fundamental forms, then up to a rigid motion, $f(M)$ is one of the following:

(i)

$f(T^2) \subset \mathbb{C}\mathrm{P}^2$ with $\kappa = \frac{1}{8}$, $S = 2$, $\cos\theta = 0$ and $K = 0$, or

(ii)

$f(S^2) \subset \mathbb{C}\mathrm{P}^4$ with $\kappa = 0$, $S = \frac{4}{3}$, $\cos\theta = 0$ and $K = \frac{1}{3}$, or

(iii)

$f(S^2) \subset \mathbb{C}\mathrm{P}^2$ with $\kappa = 0$, $S = 0$, $\cos\theta = 0$ and $K = 1$, or

(iv)

$f(S^2) \subset \mathbb{C}\mathrm{P}^3$ with $\kappa = 0$, $S = \frac{48}{49}$, $\cos\theta = \frac{1}{7}$ and $K = \frac{4}{7}$, or

(v)

$f(S^2) \subset \mathbb{C}\mathrm{P}^3$ with $\kappa = 0$, $S = \frac{48}{49}$, $\cos\theta = -\frac{1}{7}$ and $K = \frac{4}{7}$.

Here K is the Gaussian curvature of M.
Reviewed by Josué Meléndez

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This list reflects references listed in the original paper as accurately as possible with no attempt to correct error.

COLEGIO DE BACHILLERES

Plantel 7
Iztapalapa

RECONOCIMIENTO

A:

Dr. Josué Meléndez Sánchez

Por haber impartido la conferencia:

¿Qué es la geometría diferencial?

[redacted]

Mtro. Armando Polina Saldivar

CDMX a 11 de junio de 2025

Subsecretaria de Ciencia y Humanidades
Dirección General de Becas y Apoyos a la Comunidad Científica y Humanística

Ciudad de México, a 25 de junio de 2025

Numero de CVU: 165260

DR. JOSUE MELENDEZ SANCHEZ
Presente

La Secretaría de Ciencia, y Humanidades, Tecnología e Innovación (SECIHTI), por mi conducto, expresa a Usted un reconocimiento por la invaluable colaboración y aportaciones que nos ha brindado como:

E v a l u a d o r

De la solicitud con número: BP-BSNAC-20250429103428899-10682108 presentada en la

Convocatoria 2025 Estancias Sabáticas Vinculadas a la Consolidación de Grupos de Investigación

Gracias a su distinguida colaboración, la SECIHTI contó con elementos sólidos de juicio que, sin lugar a duda, permitieron que los aspirantes con mayor mérito recibieran el apoyo solicitado con base en los principios de equidad, transparencia e imparcialidad que rigen el proceder de esta Secretaría.

Al reiterar mi agradecimiento, aprovecho la oportunidad para enviarle un cordial saludo.

Atentamente,

[redacted]

Dra. Liza Elena Aceves López

Directora General de Becas y Apoyos a la Comunidad Científica y Humanística

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Some remarks on warped products

--Manuscript Draft--

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Abstract:	We establish an integral inequality for the Ricci curvature of some class of warped products $\widebar{M}=M\times_fN$, where the equality holds if and only if $\widebar{M}$ is simply a Riemannian product. We also give a sufficient condition for the intersection of a warped product $M=\mathbb{R}\times_fP$ with a totally geodesic hypersurface N in an arbitrary Riemannian space to be a totally geodesic slice of M.

SOME REMARKS ON WARPED PRODUCTS

JOSUÉ MELÉNDEZ AND EDUARDO RODRÍGUEZ-ROMERO

Abstract. We establish an integral inequality for the Ricci curvature of some class of warped products $\overline{M} = M \times_f N$, where the equality holds if and only if $\overline{M}$ is simply a Riemannian product. We also give a sufficient condition for the intersection of a warped product $M = \mathbb{R} \times_f P$ with a totally geodesic hypersurface N in an arbitrary Riemannian space to be a totally geodesic slice of M.

1. Introduction and main results

One of the most useful extensions of the Cartesian product is the notion of warped product, first defined in [2, Section 7]. As a generalization of the simple product of Riemannian manifolds, the warped products have given rise to a large family of interesting and useful examples of Riemannian manifolds, including some fundamental ones in general relativity (see [8] as a reference). Naturally, the study of submanifolds in warped products also has been a very active field in differential geometry (just to mention some works about it, see [1,3] and the references therein).

Recently in [5], Meléndez and Hernández obtained an integral inequality of the Ricci curvature for the warped product $S^1 \times_f N$, which gives a characterization of the simple product $S^1 \times N$. More precisely, they proved the following:

Theorem 1. *Let N be a compact Riemannian manifold. Consider the warped product $\overline{M} = S^1 \times_f N$ and let ∂_t be the coordinate vector field on S^1. Then*

$$\int_{\overline{M}} \operatorname{Ric}(\partial_t, \partial_t)\, \mathrm{d}\overline{M}^n \geq 0,$$

where Ric *denotes the Ricci curvature on $\overline{M}$. Moreover, equality holds if and only if $\overline{M}$ is simply a Riemannian product.*

Following the main idea of the proof (see Theorem 4 in [5]), we establish the next integral inequality for a more general warped product $M \times_f N$.

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Theorem 2. *Let M and N be Riemannian manifolds, where M is Ricci flat and* $\dim(M) = m$. *If the warped product $\overline{M} = M \times_f N$ is compact, then*

$$\sum_{k=1}^{m} \int_{\overline{M}} \mathrm{Ric}(E_k, E_k)\, \mathrm{d}\overline{M} \geq 0,$$

where Ric *denotes the Ricci curvature on $\overline{M}$ and $\{E_1, \ldots, E_m\}$ is a frame on M. Moreover, equality holds if and only if $\overline{M}$ is a Riemannian product.*

Example 1. Consider the warped product

$$\overline{M}^n = T^{n-k} \times_f S^k$$

of the $(n-k)$-dimensional flat torus $T^{n-k} = S^1 \times \cdots \times S^1$ and the standard k-sphere S^k. Since Ricci tensor of T^{n-k} is identically zero, we have

$$\sum_{k=1}^{n-k} \int_{\overline{M}} \mathrm{Ric}(\partial_{t_k}, \partial_{t_k})\, \mathrm{d}\overline{M} \geq 0,$$

where $\partial_{t_k} \in TS^1$. In particular, if $k = n-1$ we obtain

$$\overline{M} = S^1 \times_f S^{n-1}$$

It follows from Theorem 2 that

$$\int_{\overline{M}} \mathrm{Ric}(\partial_t, \partial_t)\, \mathrm{d}\overline{M} \geq 0,$$

where $\partial_t \in TS^1$. Moreover, equality holds if and only if $\overline{M}$ is isometric to a Clifford hypersurface $S^1 \times S^{n-1}$ (see Theorem 4 in [5] and Theorem 4.1 in [7]).

Now consider the class of warped products

$$M^n = \mathbb{R} \times_f P^{n-1},$$

where P is a Riemannian manifold. Given $t \in \mathbb{R}$, the *slice* Σ_t is defined as the hypersurface $\Sigma_t = \{t\} \times P$ of M. Note that Σ_t is a totally umbilical hypersurface with constant mean curvature $\mathcal{H}(t) = f'(t)/f(t)$. Recall that the *height function* $h\colon \Sigma \to \mathbb{R}$ of a hypersurface Σ of M is defined by $h(p) = \pi_{\mathbb{R}}(p)$, $p \in \Sigma$, where $\pi_{\mathbb{R}}\colon M \to \mathbb{R}$ is the projection onto the first factor.

For the next results we need the concept of *normal submanifolds* (see Section 4 in [6] for more details).

Definition 1. Let M and N be submanifolds of a Riemannian manifold $\overline{M}$. We say that M and N are **normal submanifolds** in $\overline{M}$ if

- **(a)** $M \cap N$ is a submanifold of $\overline{M}$.
- **(b)** $T_pM \cap T_p^{\perp}(M \cap N) \subset T_p^{\perp}N$ for all $p \in M \cap N$.

Remark 1. Condition (b) allows us to interchange the roles of M and N, meaning that (b) is equivalent to $T_pN \cap T_p^{\perp}(M\cap N) \subset T_p^{\perp}M$ for all $p \in M\cap N$. In addition, it is not difficult to verify that if M and N are hypersurfaces, where η and ξ are the respective unit normal vector fields, then normality is equivalent to have $\phi = \measuredangle(\eta, \xi) = \frac{\pi}{2}$ along $M \cap N$. In this case M and N intersect transversally, thus $M \cap N$ is a submanifold of $\overline{M}$ of codimension 2.

Let $M = \mathbb{R} \times_f P^{n-1}$. We now consider an isometric immersion $F : M \to \overline{M}^{n+1}$ into a Riemannian manifold $\overline{M}^{n+1}$. If N is a totally geodesic hypersurface of $\overline{M}$, the next result gives us sufficient conditions for $M \cap N$ to be a totally geodesic slice of M, provided that M and N are normal hypersurfaces in $\overline{M}$.

Theorem 3. *Let $\overline{M}^{n+1}$ be a Riemannian manifold, N a totally geodesic hypersurface of $\overline{M}$ and $M^n = \mathbb{R} \times_f P^{n-1}$ a warped product hypersurface in $\overline{M}$ with $\mathcal{H}'(t) \geq 0$, where $\mathcal{H}(t) = f'(t)/f(t)$. Suppose that M and N are normal hypersurfaces such that $\Sigma^{n-1} = M \cap N$ is a complete parabolic submanifold of $\overline{M}$ with Ricci curvature bounded from below and bounded height function $h\colon \Sigma \to \mathbb{R}$. Then Σ is a totally geodesic slice of M.*

The proof of Theorem 3 is an application of the Omori–Yau maximum principle to h. The next corollary is a more practical version of the above theorem.

Corollary 1. *Let $\overline{M}^{n+1}$ be a Riemannian manifold, N a totally geodesic hypersurface of $\overline{M}$ and $M^n = \mathbb{R} \times_f P^{n-1}$ a warped product hypersurface in $\overline{M}$ with $\mathcal{H}'(t) \geq 0$. Suppose that M and N are normal hypersurfaces such that $\Sigma^{n-1} = M \cap N$ is compact. Then Σ is a totally geodesic slice of M and P is compact.*

Example 2. Consider a rotation hypersurface M^n in the Euclidean space $\mathbb{R}^{n+1}$. We denote by $(x_1, \ldots, x_{n+1})$ the coordinates in $\mathbb{R}^{n+1}$, and we parametrize M by

$$\varphi(t, s_1, \ldots, s_{n-1}) = \big(t, f(t)\Phi(s_1, \ldots, s_{n-1})\big),$$

where $(t, f(t))$ is the profile curve of M, with $f(t) > 0$ for all t, and Φ is a parametrization of the unit sphere S^{n-1}.

Observe that M has the warped product metric

$$\langle\, , \rangle_M = dt^2 + f(t)^2 d\sigma_{n-1}^2$$

where $d\sigma_{n-1}^2$ denotes the standard round metric of the sphere S^{n-1}.

If we assume that $\mathcal{H}'(t) \geq 0$, then, as a direct application of Corollary 1, we see that the parallels

$$\Sigma^{n-1} = M \cap N = \{t_0\} \times f(t_0)\, S^{n-1}$$

are totally geodesic in M if the intersection of M with the horizontal hyperplane

$$N = \big\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = t_0\big\}$$

is normal, which only happens when $f'(t_0) = 0$.

We organize the paper as follows. First we give some general preliminaries in Section 2. Next we present the proofs of Theorems 2 and 3 in Sections 3 and 4, respectively, where we give some notation and auxiliary results used in each proof. Finally, in Section 5 we address some properties of the intersection of submanifolds.

2. General preliminaries

Throughout the manuscript $\overline{M}$ will always denote the ambient space, which is a Riemannian manifold with metric $\langle\, ,\rangle$ and Riemannian connection ∇. M and N also will denote Riemannian manifolds with metrics $\langle\, ,\rangle_M$ and $\langle\, ,\rangle_N$, and Riemannian connections ∇^M and ∇^N, respectively.

Let $f\colon M \to \mathbb{R}^+$ be a smooth function. The *warped product* $\overline{M} = M \times_f N$ is the product manifold $M \times N$ endowed with the warped metric

$$\langle X, Y\rangle = \langle d\pi_M(X), d\pi_M(Y)\rangle_M + (f \circ \pi_M)^2 \langle d\pi_N(X), d\pi_N(Y)\rangle_N ,$$

where π_M and π_N are the projections of $\overline{M}$ onto the corresponding factor. In a compact way we write

$$\langle\, ,\rangle = \langle\, ,\rangle_M + f^2 \langle\, ,\rangle_N .$$

The function f is called the *warping function* of $\overline{M}$. Notice that if f is constant, then $M \times_f N$ is the Riemannian product $M \times N$ where N has the metric $f^2 \langle\, ,\rangle_N$.

Let $f\colon \overline{M} \to \mathbb{R}$ be a smooth function. We denote by $\operatorname{grad} f$ the *gradient* of f, and by $\operatorname{Hess} f$ the Hessian of f, which are defined by

$$df(X) = \langle \operatorname{grad} f, X\rangle \quad \text{and} \quad \operatorname{Hess} f(X,Y) = \langle \nabla_X(\operatorname{grad} f), Y\rangle ,$$

where X, Y are vector fields in $\overline{M}$. In addition, the *Laplacian* of f is considered with the sign convention

$$\Delta f = \operatorname{tr}(\operatorname{Hess} f),$$

where tr denotes the trace of a linear operator.

We use similar notation, but with a superscript, for the above differential operators when f is a real-valued function defined on M or N. For example, we write $\operatorname{grad}^M f$, $\operatorname{Hess}^M f$ and $\Delta^M f$.

3. Proof of Theorem 2

We denote by Ric and Ric^M the Ricci curvatures of $\overline{M}$ and M, respectively. For the proof of Theorem 2 we need a well-known fact about warped products, and given that it shares the same context with another fact that will be used in the proof of Theorem 3, we present both in the following lemma (see Chapter 7 in [8]).

Lemma 1. *Let M and N be Riemannian manifolds, with $n = \dim(N) > 1$, and let $f\colon M \to \mathbb{R}^+$ be a smooth function. Consider the warped product $\overline{M} = M \times_f N$. Let X, Y be horizontal vector fields, and let V be a vertical vector field. Then*

(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}^M(X,Y) - \frac{n}{f}\,\mathrm{Hess}^M f(X,Y)$.

(2) $\nabla_X V = \nabla_V X = \frac{X(f)}{f} V$.

We also need the next technical lemma contained in Proposition 2.3 of [4] (see also Lemma 4 of [5]).

Lemma 2. *Let M and N be Riemannian manifolds, with $n = \dim(N)$, and let $f\colon M \to \mathbb{R}^+$ be a smooth function. Consider the warped product $\overline{M} = M \times_f N$ and $u \in C^\infty(\overline{M})$. Then*

$$\Delta u = \Delta^M u + \frac{n}{f}\left\langle \mathrm{grad}^M f, \mathrm{grad}^M u\right\rangle_M + \frac{1}{f^2}\Delta^N u.$$

Proof of Theorem 2. Let $\{E_1, \ldots, E_m\}$ be an orthonormal frame in M. Since M is Ricci flat, Lemma 1 implies

$$\sum_{k=1}^m \mathrm{Ric}(E_k,E_k) = -\frac{n}{f}\sum_{k=1}^m \mathrm{Hess}^M f(E_k,E_k) = -\frac{n}{f}\Delta^M f. \tag{1}$$

If we set $u = \ln f$, we obtain

$$\begin{aligned}\Delta^M u &= \sum_{k=1}^m \left\langle \nabla^M_{E_k}(\mathrm{grad}^M u), E_k\right\rangle_M \\ &= \sum_{k=1}^m \left\langle \nabla^M_{E_k}\left(\frac{\mathrm{grad}^M f}{f}\right), E_k\right\rangle_M = \frac{1}{f}\Delta^M f - \frac{1}{f^2}\left\|\mathrm{grad}^M f\right\|_M^2.\end{aligned} \tag{2}$$

Now, by using Lemma 2,

$$\Delta u = \Delta^M u + \frac{n}{f}\left\langle \mathrm{grad}^M f, \mathrm{grad}^M u\right\rangle_M = \Delta^M u + \frac{n}{f^2}\left\|\mathrm{grad}^M f\right\|_M^2. \tag{3}$$

If we substitute (2) in (3), we find that

$$\Delta u = \frac{\Delta^M f}{f} + (n-1)\left(\frac{\left\|\mathrm{grad}^M f\right\|_M}{f}\right)^2 = \frac{\Delta^M f}{f} + (n-1)\left\|\mathcal{H}\right\|_M^2,$$

where $\mathcal{H} = \frac{1}{f}\,\mathrm{grad}^M f$. It follows from (1) that

$$n\Delta u = -\sum_{k=1}^m \mathrm{Ric}(E_k,E_k) + n(n-1)\left\|\mathcal{H}\right\|_M^2. \tag{4}$$

By the compactness of $\overline{M}$, we can integrate both sides of (4) to obtain

$$\sum_{k=1}^{m} \int_{\overline{M}} \mathrm{Ric}(E_k, E_k)\, d\overline{M} = n(n-1) \int_{\overline{M}} \|\mathcal{H}\|_M^2 \; d\overline{M} \geq 0.$$

Observe that equality holds if and only if $\mathcal{H} = 0$, which means that f is constant. $\square$

4. Proof of Theorem 3

We say that M is *parabolic* if the only subharmonic functions on M which are bounded from above are the constant ones. Explicitly stated, this means that if $u \in C^2(M)$ is such that $\Delta^M u \geq 0$ and $\sup_M u < \infty$, then u must be constant. Being parabolic is equivalent to have that the only superharmonic functions on M which are bounded from below are the constant ones, or explicitly, if $u \in C^2(M)$ satisfies $\Delta^M u \leq 0$ and $\inf_M u > -\infty$, then u must be constant.

It is well known that every compact manifold is parabolic. In particular, any sphere S^n is parabolic. On the other hand, the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n = 1, 2$. To see that $\mathbb{R}^n$ is not parabolic for $n \geq 3$, it is sufficient to give an explicit example of a positive non constant superharmonic function, like the map

$$u(x) = (1 + \|x\|^2)^{-\frac{n-2}{2}}.$$

For the proof of Theorem 3 we use the following well-known principle due to H. Omori and S. T. Yau (see [9]).

Theorem 4. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Consider $u \in C^2(M)$ that is bounded from below on M. Then there exists a sequence $\{p_j\}$ in M such that*

$$\lim_{j\to\infty} u(p_j) = \inf_M u, \quad \left\|\mathrm{grad}^M u(p_j)\right\| < \frac{1}{j}, \quad \Delta^M u(p_j) > -\frac{1}{j}. \tag{5}$$

Let N be a submanifold of $\overline{M}$. Let us denote by $B_N^{\overline{M}}$ the second fundamental form of N in $\overline{M}$, this means that

$$\overline{\nabla}_X Y = \nabla_X^N Y + B_N^{\overline{M}}(X, Y), \quad X, Y \in \mathfrak{X}(N).$$

If $n = \dim(N)$, the mean curvature vector of N in $\overline{M}$ is given by

$$\mathbf{H}_N^{\overline{M}} = \frac{1}{n} \sum_{i=1}^{n} B_N^{\overline{M}}(E_i, E_i),$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on N. A submanifold N is said to be totally umbilical if

$$B_N^{\overline{M}}(X, Y) = \langle X, Y\rangle_N \mathbf{H}_N^{\overline{M}}$$

for every $X, Y \in \mathfrak{X}(N)$.

Let P^{n-1} be a Riemannian manifold with metric $\langle\,,\rangle_P$, and consider the warped product $M^n = \mathbb{R} \times_f P$ with its warped metric $\langle\,,\rangle = dt^2 + f^2 \langle\,,\rangle_P$. Let Σ be a complete oriented hypersurface of M, where η is the unit normal vector of Σ. Then the shape operator of Σ respect to η is given by

$$AX = -\nabla_X \eta, \quad \text{where } X \in \mathfrak{X}(M).$$

We also need to compute the Laplacian of the height function $h\colon \Sigma \to \mathbb{R}$ (see Proposition 2.1 of [1]). For the reader's convenience we include a detailed proof.

Lemma 3. *Let Σ^{n-1} be an oriented hypersurface of $M^n = \mathbb{R} \times_f P$. Then*

$$\Delta^\Sigma h = \mathcal{H}(h)\big(n-1-\left\|\operatorname{grad}^\Sigma h\right\|^2\big) + (n-1)\left\langle \partial_t, \mathbf{H}^M_\Sigma \right\rangle, \tag{6}$$

where $\mathbf{H}^M_\Sigma$ is the mean curvature vector of Σ respect to M and ∂_t is the coordinate vector field of the first factor of M.

Proof. First note that the gradient of the projection $\pi_1\colon M \to \mathbb{R}$ is given by $\operatorname{grad}^M \pi_1 = \partial_t$. Thus the gradient of the height function $h(p) = \pi_1(p)$ is given by

$$\operatorname{grad}^\Sigma h = (\operatorname{grad}^M \pi_1)^\top = \partial_t^\top = \partial_t - \langle \partial_t, \eta\rangle \eta, \tag{7}$$

where $(\cdot)^\top$ denotes the projection over $T\Sigma$. On the other hand, we know that each $X \in \mathfrak{X}(M)$ can be decomposed as $X = \langle X, \partial_t\rangle \partial_t + V$, where $V = X - \langle X, \partial_t\rangle \partial_t$. It follows from item (2) of Lemma 1 that

$$\nabla^M_X \partial_t = \langle X, \partial_t\rangle \cancelto{0}{\nabla_{\partial_t}\partial_t} + \nabla_V \partial_t = \mathcal{H}(t)V = \mathcal{H}(t)(X - \langle X, \partial_t\rangle \partial_t), \tag{8}$$

where $\mathcal{H}(t) = f'(t)/f(t)$. Using (7) and (8) we deduce that

$$\nabla^M_X(\operatorname{grad}^\Sigma h) = \nabla_X(\partial_t - \langle \partial_t, \eta\rangle \eta) = \mathcal{H}(t)(X - \langle X, \partial_t\rangle \partial_t) - X\langle \partial_t, \eta\rangle \eta - \langle \partial_t, \eta\rangle \nabla_X \eta.$$

If we consider $X \in \mathfrak{X}(\Sigma)$ in the above equation, by projecting over $T\Sigma$ we get

$$\begin{aligned}
\nabla^\Sigma_X(\operatorname{grad}^\Sigma h) &= \big(\nabla^M_X(\operatorname{grad}^\Sigma h)\big)^\top \\
&= \mathcal{H}(h)\big(X - \langle X, \partial_t\rangle \partial_t^\top\big) - \langle \partial_t, \eta\rangle (\nabla_X \eta)^\top \\
&= \mathcal{H}(h)\big(X - \left\langle X, \operatorname{grad}^\Sigma h\right\rangle \operatorname{grad}^\Sigma h\big) + \langle \partial_t, \eta\rangle AX.
\end{aligned}$$

Therefore, if we consider an orthonormal frame $\{E_i\}$ on Σ, we conclude that

$$\begin{aligned}\Delta^\Sigma h &= \sum_{i=1}^{n-1} \left\langle \nabla^\Sigma_{E_i}(\operatorname{grad}^\Sigma h), E_i \right\rangle \\ &= \sum_{i=1}^{n-1} \Big(\mathcal{H}(h)\big(1 - \left\langle E_i, \operatorname{grad}^\Sigma h \right\rangle^2\big) + \langle \partial_t, \eta \rangle \langle AE_i, E_i \rangle \Big) \\ &= \mathcal{H}(h)\big(n - 1 - \left\|\operatorname{grad}^\Sigma h\right\|^2\big) + (n-1)\left\langle \partial_t, \mathbf{H}^M_\Sigma \right\rangle.\end{aligned}$$

□

Proof of Theorem 3. Let $M = \mathbb{R} \times_f P$ be a warped product hypersurface and N a totally geodesic hypersurface of $\overline{M}$, and assume that M and N are normal hypersurfaces. Then Corollary 2 in Section 5 implies

$$\mathbf{H}^M_\Sigma = (\mathbf{H}^{\overline{M}}_N)^\top = 0 \quad \text{on } \Sigma = M \cap N,$$

where $(\cdot)^\top$ denotes the projection over TM. Then equation (6) yields

$$\Delta^\Sigma h = \mathcal{H}(h) \left(n - 1 - \left\|\operatorname{grad}^\Sigma h\right\|^2 \right). \tag{9}$$

By the Omori-Yau maximum principle (5) we know that there exists a sequence $\{p_j\}$ in Σ such that

$$\lim_{j\to\infty} h(p_j) = h_*, \quad \left\|\operatorname{grad}^\Sigma h(p_j)\right\|^2 < \frac{1}{j^2}, \quad \Delta^\Sigma h(p_j) > -\frac{1}{j},$$

where $h_* = \inf_\Sigma h$, and by the previous equation we obtain

$$\mathcal{H}(h(p_j))(n - 1 - \left\|\operatorname{grad}^\Sigma h(p_j)\right\|^2) > -\frac{1}{j},$$

which reduces to $\mathcal{H}(h_*) \geq 0$ when $j \to \infty$, and since $\mathcal{H}' \geq 0$, we get

$$\mathcal{H}(h) \geq 0 \quad \text{on } \Sigma. \tag{10}$$

Let η be the unit normal vector of Σ in M. As $\operatorname{grad}^\Sigma h = \partial_t - \langle \partial_t, \eta \rangle \eta$, it follows

$$\left\|\operatorname{grad}^\Sigma h\right\|^2 = 1 - \langle \partial_t, \eta \rangle^2.$$

Therefore

$$(n-1) - \left\|\operatorname{grad}^\Sigma h\right\|^2 = (n-2) + \langle \partial_t, \eta \rangle^2 \geq 0.$$

By combining (9) and (10), we conclude $\Delta^\Sigma h \geq 0$ on Σ, but given that $\sup_\Sigma h < \infty$, the parabolicity of Σ implies that h is constant with $\mathcal{H}(h) = 0$, and consequently Σ is a totally geodesic slice of M. □

5. Some properties of intersection of submanifolds

In the proof of Theorem 3 we have used the last result of this section, which deals with a property of the normal intersection of two hypersurfaces in a Riemannian ambient. This auxiliary result can be seen as a consequence of a general pattern concerning the intersection of submanifolds of arbitrary codimension.

Proposition 1. *Let M and N be submanifolds of a Riemannian manifold $\overline{M}$ such that $\Sigma = M \cap N$ is a submanifold of $\overline{M}$. If N is totally umbilical in $\overline{M}$, then*

$$\mathbf{H}_\Sigma^M = (\mathbf{H}_\Sigma^N)^\top + (\mathbf{H}_N^{\overline{M}})^\top \quad \textit{along } \Sigma,$$

where $(\cdot)^\top$ is the projection over TM.

Proof. Recall that for all $X, Y \in \mathfrak{X}(\Sigma)$, we have the decomposition

$$B_\Sigma^{\overline{M}}(X,Y) = B_\Sigma^M(X,Y) + B_M^{\overline{M}}(X,Y) = B_\Sigma^N(X,Y) + B_N^{\overline{M}}(X,Y).$$

Therefore, if $k = \dim(\Sigma)$ and $\{E_1, \ldots, E_k\}$ is a local orthonormal frame in Σ, we deduce from de umbilicity of N in $\overline{M}$ that

$$\begin{aligned} k\mathbf{H}_\Sigma^M &= \sum_{i=1}^k B_\Sigma^M(E_i, E_i) \\ &= \sum_{i=1}^k B_\Sigma^N(E_i, E_i) + \sum_{i=1}^k B_N^{\overline{M}}(E_i, E_i) - \sum_{i=1}^k B_M^{\overline{M}}(E_i, E_i) \\ &= k\mathbf{H}_\Sigma^N + k\mathbf{H}_N^{\overline{M}} - \sum_{i=1}^k B_M^{\overline{M}}(E_i, E_i), \end{aligned}$$

and by projecting the last equation to TM we get the desired formula. □

When M and N are hypersurfaces and do not intersect "tangentially", we obtain:

Corollary 2. *Let M and N be oriented hypersurfaces of $\overline{M}$, where η and ξ are their respective unit normal vectors, and suppose that $M \cap N \neq \emptyset$. If N is totally umbilical in $\overline{M}$ and the angle $\phi = \angle(\eta, \xi) \in (0, \pi)$, then*

$$\mathbf{H}_\Sigma^M = \left(H_\Sigma^N \cos(\phi) \pm H_N^{\overline{M}} \sin(\phi)\right) \eta_* \quad \textit{along } \Sigma = M \cap N,$$

where η_* is the unit normal vector of Σ respect to M, and H_Σ^N and $H_N^{\overline{M}}$ are the scalar mean curvatures of $\Sigma \subset N$ and $N \subset \overline{M}$, respectively. In particular, when M and N are normal hypersurfaces in $\overline{M}$, we have

$$\mathbf{H}_\Sigma^M = (\mathbf{H}_N^{\overline{M}})^\top = H_N^{\overline{M}} \xi$$

and Σ is totally umbilical in M.

Proof. As M and N are hypersurfaces and $\phi \in (0, \pi)$, necessarily M and N intersect transversally, so Σ is a submanifold of $\overline{M}$ of codimension 2 (see Remark 1).

Let η_* and ξ_* be the unit normal vectors of Σ respect to η and ξ, respectively, meaning that $\{\eta_*, \eta\}$ and $\{\xi_*, \xi\}$ are positively oriented orthonormal bases of the plane $T_p^{\perp}\Sigma$ for all $p \in \Sigma$. Consequently the frame $\{\xi_*, \xi\}$ is obtained by rotating the frame $\{\eta_*, \eta\}$ an angle ϕ, and depending on the position of the frames, the rotation can be clockwise or counterclockwise. Therefore we can write

$$\xi_* = \cos(\phi)\eta_* \pm \sin(\phi)\eta, \quad \xi = \mp \sin(\phi)\eta_* + \cos(\phi)\eta,$$

which implies

$$(\mathbf{H}_\Sigma^N)^\top = (H_\Sigma^N \xi_*)^\top = H_\Sigma^N \cos(\phi)\eta_*, \quad (\mathbf{H}_N^{\overline{M}})^\top = (H_N^{\overline{M}} \xi)^\top = \mp H_N^{\overline{M}} \sin(\phi)\eta_*.$$

Therefore, from Proposition 1 we obtain

$$\mathbf{H}_\Sigma^M = (\mathbf{H}_\Sigma^N)^\top + (\mathbf{H}_N^{\overline{M}})^\top = \Big(H_\Sigma^N \cos(\phi) \mp H_N^{\overline{M}} \sin(\phi) \Big) \eta_*.$$

If M and N are normal hypersurfaces in $\overline{M}$ ($\phi \equiv \frac{\pi}{2}$), then

$$\mathbf{H}_\Sigma^M = (\mathbf{H}_N^{\overline{M}})^\top = \mp H_N^{\overline{M}} \eta_* = H_N^{\overline{M}} \xi.$$

□

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